

NOTE ON PFAFFIANS CONNECTED WITH THE DIFFERENCE-PRODUCT.

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(1) In the year 1886 Torelli announced* without proof the identity

$$\begin{vmatrix} (a_1 - a_2)^{2m-1} & (a_1 - a_3)^{2m-1} & \dots & (a_1 - a_{2m})^{2m-1} \\ (a_2 - a_3)^{2m-1} & (a_2 - a_4)^{2m-1} & \dots & (a_2 - a_{2m})^{2m-1} \\ \dots & \dots & \dots & \dots \\ (a_{2m-1} - a_{2m})^{2m-1} & & & \end{vmatrix} \\ = (-1)^{\frac{1}{2}m(m+1)} (2m-1)_1 (2m-1)_2 \dots (2m-1)_{m-1} \begin{vmatrix} a_1^0 & a_1^1 & \dots & a_1^{2m-1} \\ a_2^0 & a_2^1 & \dots & a_2^{2m-1} \\ \dots & \dots & \dots & \dots \\ a_{2m}^0 & a_{2m}^1 & \dots & a_{2m}^{2m-1} \end{vmatrix},$$

where $(2m-1)_1, (2m-2)_2, \dots$ are the coefficients of the expanded binomials. Probably the easiest way to establish it is to make it dependent on Zehfuss' double-alternant theorem

$$\begin{vmatrix} (a_1 + b_1)^{n-1} & (a_2 + b_2)^{n-1} & \dots & (a_n + b_n)^{n-1} \end{vmatrix} \\ = (-1)^{\frac{1}{2}n(n-1)} (n-1)_1 (n-1)_2 \dots (n-1)_{n-1} \cdot \xi^{\frac{1}{2}}(a_1, \dots, a_n) \cdot \xi^{\frac{1}{2}}(b_1, \dots, b_n)$$

Putting in this $n = 2m$, $b_r = -a_r$ the determinant on the left becomes zero-axial skew, the binomial-coefficients become even in number with the first half the same as the second, the two difference-products become identical save as to sign, and the additional sign-factor thus introduced is the same as that already there. The extraction of the square root is thus a simple matter.

(2) As a first step forward from this let us examine the consequence of putting in it $a_{2m} = 0$. On the right-hand side the determinant, whether we view it directly as such or indirectly as the difference-product, is seen to be naturally separable into two parts, namely,

$$(-1)^{2m-1} a_1 a_2 \dots a_{2m-1} \quad \text{and} \quad \begin{vmatrix} a_1^0 & a_1^1 & \dots & a_1^{2m-2} \\ a_2^0 & a_2^1 & \dots & a_2^{2m-2} \\ \dots & \dots & \dots & \dots \\ a_{2m-1}^0 & a_{2m-1}^1 & \dots & a_{2m-1}^{2m-2} \end{vmatrix};$$

but on the left-hand side there is no corresponding appearance of resolvable

* *Giornale di Mat.*, xxiv, p. 377. See also Muir's textbook (1882), pp. 206, 240, ex 15.

bility. On examination, however, it is found that the elements of the last frame-line, namely,

$$a_1^{2m-1}, a_2^{2m-1}, \dots, a_{2m-1}^{2m-1}$$

may be all changed into 1 if we simultaneously annex

$$a_1 a_2 \dots a_{2m-1}$$

to the Pfaffian as a factor; for example, when $m = 2$ we have

$$\begin{vmatrix} (a_1 - a_2)^3 & (a_1 - a_3)^3 & a_1^3 \\ (a_2 - a_3)^3 & a_2^3 & \\ a_3^3 & & 1 \end{vmatrix} = \begin{vmatrix} (a_1 - a_2)^3 & (a_1 - a_3)^3 & 1 \\ & (a_2 - a_3)^3 & 1 \\ & & 1 \end{vmatrix} \cdot a_1 a_2 a_3.$$

On the removal therefore of the common factor $a_1 a_2 \dots a_{2m-1}$ we obtain

$$\begin{vmatrix} (a_1 - a_2)^{2m-1} & (a_1 - a_3)^{2m-1} & \dots & (a_1 - a_{2m-1})^{2m-1} & 1 \\ & (a_2 - a_3)^{2m-1} & \dots & (a_2 - a_{2m-1})^{2m-1} & 1 \\ & & \dots & & \\ & & & (a_{2m-2} - a_{2m-1})^{2m-1} & 1 \\ & & & & 1 \end{vmatrix} \\ = (-1)^{\frac{1}{2}(m+1)(m+2)} \cdot (2m-1)_1 (2m-1)_2 \dots (2m-1)_{m-1} \cdot \begin{vmatrix} a_1^0 & a_2^1 & \dots & a_{2m-1}^{2m-2} \end{vmatrix}$$

(3) It should be noted in passing that there is a theorem in determinants corresponding to the lemma used in the foregoing, namely, when m is 2,

$$\begin{vmatrix} (a_1 + b_1)^3 & (a_1 + b_2)^3 & (a_1 + b_3)^3 & a_1^3 \\ (a_2 + b_1)^3 & (a_2 + b_2)^3 & (a_2 + b_3)^3 & a_2^3 \\ (a_3 + b_1)^3 & (a_3 + b_2)^3 & (a_3 + b_3)^3 & a_3^3 \\ b_1^3 & b_2^3 & b_3^3 & . \end{vmatrix} = a_1 a_2 a_3 \cdot b_1 b_2 b_3 \begin{vmatrix} (a_1 + b_1)^3 & (a_1 + b_2)^3 & (a_1 + b_3)^3 & 1 \\ (a_2 + b_1)^3 & (a_2 + b_2)^3 & (a_2 + b_3)^3 & 1 \\ (a_3 + b_1)^3 & (a_3 + b_2)^3 & (a_3 + b_3)^3 & 1 \\ 1 & 1 & 1 & . \end{vmatrix}$$

the determinant on the left being equal to

$$\begin{vmatrix} a_1^3 & 3a_1^2 & 3a_1 & 1 \\ a_2^3 & 3a_2^2 & 3a_2 & 1 \\ a_3^3 & 3a_3^2 & 3a_3 & 1 \\ . & . & . & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & b_1 & b_1^2 & b_1^3 \\ 1 & b_2 & b_2^2 & b_2^3 \\ 1 & b_3 & b_3^2 & b_3^3 \\ 1 & . & . & . \end{vmatrix}$$

and the determinant on the right equal to

$$\begin{vmatrix} a_1^3 & 3a_1^2 & 3a_1 & 1 \\ a_2^3 & 3a_2^2 & 3a_2 & 1 \\ a_3^3 & 3a_3^2 & 3a_3 & 1 \\ 1 & . & . & . \end{vmatrix} \cdot \begin{vmatrix} 1 & b_1 & b_1^2 & b_1^3 \\ 1 & b_2 & b_2^2 & b_2^3 \\ 1 & b_3 & b_3^2 & b_3^3 \\ . & . & . & 1 \end{vmatrix}$$

Indeed a series of identities can be obtained in this way by giving different positions to the units in the last rows of the multipliers; and from this series can be deduced another, namely, in the case of the third order,

$$\begin{vmatrix} (\alpha - \beta)^3 & (\alpha - \gamma)^3 & \alpha^s \\ & (\beta - \gamma)^3 & \beta^s \\ & & \gamma^s \end{vmatrix}$$

has the equivalents

$$3 \mid \alpha^0 \beta^1 \gamma^2 \mid, \mid \alpha^0 \beta^1 \gamma^3 \mid, \mid \alpha^0 \beta^2 \gamma^3 \mid, \mid 3 \mid \alpha^1 \beta^2 \gamma^3 \mid$$

when s is 0, 1, 2, 3 respectively.

(4) In the double-alternant theorem above used the index of the power to which each binomial is raised is 1 less than the number of variables in each of the two sets. When the index is the same as the number of variables, that is to say, when the determinant is

$$\mid (a_1 + b_1)^n (a_2 + b_2)^n \dots (a_n + b_n)^n \mid$$

no useful outcome is obtained by putting $b_r = -a_r$; for, when n is odd, the resulting zero-axial skew determinant being of odd degree vanishes, and when n is even the resulting determinant is not skew.

When the index is 1 more than the number of variables, one case is unfruitful and the other not. For if n be odd the determinant

$$\mid (a_1 + b_1)^{n+1} (a_2 + b_2)^{n+1} \dots (a_n + b_n)^{n+1} \mid$$

is not skew, and if n be even it is both skew and of even order. Calling the three alternants spoken of

$$D_{n:n-1}, \quad D_{n:n}, \quad D_{n:n+1}$$

we may note the general conclusion that $D_{n;p}$ for the substitution $b_r = -a_r$ becomes the square of a Pfaffian when n is even and p odd.

(5) The theorem that is available for our purpose in regard to $D_{n;n+1}$ is one that appeared in the *Transac. R. Soc. S. Africa*, iii (1912), p. 182. For $n = 4$ it is

$$\frac{D_{4;5}}{\zeta_1^{\frac{1}{2}} \zeta_2^{\frac{1}{2}}} = \begin{vmatrix} . & 1 & -\Sigma b_1 & \Sigma b_1 b_2 & -\Sigma b_1 b_2 b_3 & b_1 b_2 b_3 b_4 \\ 1 & . & . & . & -5 & -\Sigma a_1 \\ -\Sigma a_1 & . & . & -10 & . & \Sigma a_1 a_2 \\ \Sigma a_1 a_2 & . & -10 & . & . & -\Sigma a_1 a_2 a_3 \\ -\Sigma a_1 a_2 a_3 & -5 & . & . & . & a_1 a_2 a_3 a_4 \\ a_1 a_2 a_3 a_4 & -\Sigma b_1 & \Sigma b_1 b_2 & -\Sigma b_1 b_2 b_3 & b_1 b_2 b_3 b_4 & . \end{vmatrix}$$

where $\zeta_1^{\frac{1}{2}}, \zeta_2^{\frac{1}{2}}$ stand for the difference-products of the a 's and b 's respectively. When in this b_r is put $= -a_r$ and the even-ordered rows have their signs changed, the right-hand member becomes, like $D_{4;5}$, zero-axial skew, and we deduce

$$\begin{vmatrix} (a_1 - a_2)^5 & (a_1 - a_3)^5 & (a_1 - a_4)^5 \\ & (a_2 - a_3)^5 & (a_2 - a_4)^5 \\ & & (a_3 - a_4)^5 \end{vmatrix} = \zeta^{\frac{1}{2}} \begin{vmatrix} 1 & \Sigma a_1 & \Sigma a_1 a_2 & \Sigma a_1 a_2 a_3 & a_1 a_2 a_3 a_4 \\ & . & . & 5 & \Sigma a_1 \\ & & -10 & . & \Sigma a_1 a_2 \\ & & & . & \Sigma a_1 a_2 a_3 \\ & & & & a_1 a_2 a_3 a_4 \end{vmatrix}.$$

This, for shortness' sake, may be written

$$f_{4;5} = \zeta^{\frac{1}{2}} \begin{vmatrix} 1 & c_1 & c_2 & c_3 & c_4 \\ & . & . & 5 & c_1 \\ & & -10 & . & c_2 \\ & & & . & c_3 \\ & & & & c_4 \end{vmatrix};$$

and with the same notation the next case is

$$f_{6;7} = \zeta^{\frac{1}{2}} \begin{vmatrix} 1 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ & . & . & . & . & 7 & c_1 \\ & & . & . & -21 & . & c_2 \\ & & & 35 & . & . & c_3 \\ & & & & . & . & c_4 \\ & & & & & . & c_5 \\ & & & & & & c_6 \end{vmatrix},$$

the general theorem being thus evident.

(6) On account of the number of their zero elements the Pfaffians on the right may be condensed into the forms

$$\begin{vmatrix} -10 & -5c_1 & c_2 \\ & -5c_2 & c_3 \\ & & 6c_4 - c_1c_3 \end{vmatrix}, \quad \begin{vmatrix} . & . & -21 & -7c_1c_2 \\ & 35 & . & -7c_2c_3 \\ & & . & -7c_3c_4 \\ & & & -7c_4c_5 \\ & & & & 8c_6 - c_1c_5 \end{vmatrix}$$

where, however, the law of formation is not quite so evident. If we divide every element outside the last frame-line by 5 in the one case and by 7 in the other, we are thereby removing the factors 5, 7², respectively. This suggests a third form, which in the next case is

$$\begin{vmatrix} 9^3 & . & . & . & -4 & -c_1c_2 \\ & . & . & 9^{\frac{1}{2}} & . & -c_2c_3 \\ & & -14 & . & . & -c_3c_4 \\ & & & . & . & -c_4c_5 \\ & & & & . & -c_5c_6 \\ & & & & & -c_6c_7 \\ & & & & & & 10c^8 - c_1c_7 \end{vmatrix}.$$

The final expansions are

$$\begin{aligned} & -5(c_2^2 - 3c_1c_3 + 12c_4), \\ & 49(3c_3^2 - 8c_2c_4 + 20c_1c_5 - 120c_6), \\ & 13608(2c_4^2 - 5c_3c_5 + 10c_2c_6 - 35c_1c_7 + 280c_8). \end{aligned}$$

(7) At this stage we meet with a very curious relationship, the bracketed expressions just written being found to make their appearance in a totally different connection—namely, as playing a like part in the final expansions of zero-axial skew permanents of the form

$$\begin{vmatrix} + & & & & & + \\ & . & a_1 - a_2 & a_1 - a_3 & . & . \\ a_2 - a_1 & & . & a_2 - a_3 & . & . \\ a_3 - a_1 & a_3 - a_2 & & . & . & . \\ . & . & . & . & . & . \end{vmatrix}.$$

In order to show this it is at least as easy to find the expansion of the more general permanent

$$\begin{vmatrix} + & & & & & + \\ a_1 + b_1 & a_1 + b_2 & a_1 + b_3 & . & . & . \\ a_2 + b_1 & a_2 + b_2 & a_2 + b_3 & . & . & . \\ a_3 + b_1 & a_3 + b_2 & a_3 + b_3 & . & . & . \\ . & . & . & . & . & . \end{vmatrix}_n$$

This we can express as the sum of 2^n permanents with monomial elements, the best mode of arranging them being in accordance with the number of columns of a 's and the number of columns of b 's which they contain. If the elements be all a 's, the value of the permanent is

$$n! \cdot a_1 a_2 \dots a_n;$$

if there be only one column of b 's the value is

$$b_r \cdot (n-1)! \cdot \Sigma a_1 a_2 \dots a_{n-1},$$

and the sum of all such is

$$(n-1)! \cdot \Sigma b_1 \cdot \Sigma a_1 a_2 \dots a_{n-1};$$

if there be only two columns of b 's the value is

$$b_r b_s \cdot (n-2)! \cdot \Sigma a_1 a_2 \dots a_{n-2},$$

and the sum of all such is

$$2! (n-2)! \cdot \Sigma b_1 b_2 \cdot \Sigma a_1 a_2 \dots a_{n-2};$$

and so on, the last term, the $(n+1)^{\text{th}}$, being

$$n! \cdot b_1 b_2 \dots b_n,$$

and in general the n^{th} term from the end being formable from the n^{th} term from the beginning by interchanging the a 's and b 's. The expansion can therefore best be written in bifold form

$$\begin{aligned}
& n! \{a_1 a_2 \dots a_n + b_1 b_2 \dots b_n\} \\
& + (n-1)! \{ \Sigma a_1 a_2 \dots a_{n-1} \Sigma b_1 + \Sigma a_1 \Sigma b_1 b_2 \dots b_n - \\
& + (n-2)! 2! \{ \Sigma a_1 a_2 \dots a_{n-2} \Sigma b_1 b_2 + \Sigma a_1 a_2 \Sigma b_1 b_2 \dots b_{n-1} \} \\
& + \dots \dots \dots
\end{aligned}$$

the number of such double terms being $\frac{1}{2}n$ when n is even and $\frac{1}{2}(n+1)$ when n is odd, but there being also in the former case a unique single term.

(8) If now in this we put $b_r = -a_r$ and take $n = 2m$, the left-hand member becomes a zero-axial skew permanent, and the right-hand member becomes

$$\left. \begin{aligned}
& 2 \cdot (2m)! a_1 a_2 \dots a_{2m} \\
& - 2 \cdot (2m-1)! \Sigma a_1 a_2 \dots a_{2m-1} \cdot \Sigma a_1 \\
& + 2 \cdot (2m-2)! 2! \Sigma a_1 a_2 \dots a_{2m-2} \cdot \Sigma a_1 a_2 \\
& - \dots \dots \dots \\
& + (-1)^m m! m! (\Sigma a_1 a_2 \dots a_m)^2
\end{aligned} \right\} \text{ or } \left. \begin{aligned}
& 2 \cdot (2m)! c_{2m} \\
& - 2 \cdot (2m-1)! c_{2m-1} c_1 \\
& + 2 \cdot (2m-2)! 2! c_{2m-2} c_2 \\
& - \dots \dots \dots \\
& + (-1)^m (m!)^2 c_m^2
\end{aligned} \right\}$$

(9) When m is 2 this expansion is

$$\begin{aligned}
& 2 \cdot 4! c_4 - 2 \cdot 3! c_3 c_1 + (2!)^2 c_2^2 \\
& i. e. \quad 4(c_2^2 - 3c_1 c_3 + 12c_4);
\end{aligned}$$

when m is 3 it is

$$\begin{aligned}
& 2 \cdot 6! c_6 - 2 \cdot 5! c_5 c_1 + 2 \cdot 4! 2! c_4 c_2 - (3!)^2 c_3^2 \\
& i. e. \quad -12(3c_3^2 - 8c_2 c_4 + 20c_1 c_5 - 120c_6);
\end{aligned}$$

and when m is 4 it is

$$288(2c_4^2 - 5c_3 c_5 + 10c_2 c_6 - 35c_1 c_7 + 280c_8);$$

where the bracketed expressions are exactly those at the end of §6. We are thus face to face with the curious identities

$$\begin{aligned}
& \left| \begin{array}{ccc} -10 & -5c_1 & c_2 \\ & -5c_2 & c_3 \\ & & 6c_4 - c_1 c_3 \end{array} \right| = -\frac{5}{4} \left| \begin{array}{ccc} a_1 - a_2 & a_1 - a_3 & a_1 - a_4 \\ a_2 - a_1 & & a_2 - a_3 \\ a_3 - a_1 & a_3 - a_2 & & a_3 - a_4 \\ a_4 - a_1 & a_4 - a_2 & a_4 - a_3 & \end{array} \right| \\
& \left| \begin{array}{ccc} & -7c_1 & c_2 \\ 35 & & -7c_2 \\ & -7c_3 & c_4 \\ & & -7c_4 \\ & & & 8c_6 - c_1 c_5 \end{array} \right| = -\frac{4}{12} \left| \begin{array}{ccc} a_1 - a_2 & a_1 - a_3 & a_1 - a_6 \\ a_2 - a_1 & & a_2 - a_3 \\ a_3 - a_1 & a_3 - a_2 & & a_3 - a_6 \\ \dots & \dots & \dots & \dots \\ a_6 - a_1 & a_6 - a_2 & a_6 - a_3 & \dots \end{array} \right|
\end{aligned}$$

$$\begin{vmatrix} . & . & . & . & -36 & -9c_1 & c_2 \\ . & . & 84 & . & . & -9c_2 & c_3 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & -9c_6 & c_7 \\ . & . & . & . & . & 10c_8 - c_1c_7 \end{vmatrix} = \frac{189}{4} \begin{vmatrix} . & a_1 - a_2 & . & . & a_1 - a_8 \\ a_2 - a_1 & . & . & . & a_2 - a_8 \\ . & . & . & . & . \\ . & . & . & . & . \\ a_8 - a_1 & a_8 - a_2 & . & . & . \end{vmatrix} +$$

No direct mode of establishing them seems available, and the problem is not made easier by expressing them in terms of the differences of the a 's only: namely, for example,

$$\begin{vmatrix} (a_1 - a_2)^5 & (a_1 - a_3)^5 & (a_1 - a_4)^5 \\ & (a_2 - a_3)^5 & (a_2 - a_4)^5 \\ & & (a_3 - a_4)^5 \end{vmatrix} = -\frac{5}{4} \zeta_4^{\frac{1}{2}} \begin{vmatrix} . & a_1 - a_2 & a_1 - a_3 & a_1 - a_4 \\ a_2 - a_1 & . & a_2 - a_3 & a_2 - a_4 \\ a_3 - a_1 & a_3 - a_2 & . & a_3 - a_4 \\ a_4 - a_1 & a_4 - a_2 & a_4 - a_3 & . \end{vmatrix} +$$

or, generally,

$$\begin{vmatrix} (a_1 - a_2)^{2m+1} & (a_1 - a_3)^{2m+1} & . & . & (a_1 - a_{2m})^{2m+1} \\ & (a_2 - a_3)^{2m+1} & . & . & (a_2 - a_{2m})^{2m+1} \\ & & . & . & . \\ & & & . & . \\ & & & & (a_{2m-1} - a_{2m})^{2m+1} \end{vmatrix} = (-1)^{\frac{1}{2}m(m-1)} M \zeta_4^{\frac{1}{2}} \begin{vmatrix} . & a_1 - a_2 & a_1 - a_3 & . & . & a_1 - a_{2m} \\ a_2 - a_1 & . & a_2 - a_3 & . & . & a_2 - a_{2m} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{2m} - a_1 & a_{2m} - a_2 & a_{2m} - a_3 & . & . & . \end{vmatrix} +$$

where the arithmetical multiplier M is

$$\frac{1}{(m!)^2} (2m+1)_1 (2m+1)_2 \dots (2m+1)_{m-1}.$$

(10) As the permanent on the right in the preceding is the same as that which occurs in the result obtained on making our usual specialisation in $D_n; n$, we can eliminate it by division, and so obtain a relation between an *axisymmetric* determinant and a Pfaffian; for example,

$$\begin{vmatrix} . & (a_1 - a_2)^4 (a_1 - a_3)^4 (a_1 - a_4)^4 \\ (a_1 - a_2)^4 & . & (a_2 - a_3)^4 (a_2 - a_4)^4 \\ (a_1 - a_3)^4 (a_2 - a_3)^4 & . & (a_3 - a_4)^4 \\ (a_1 - a_4)^4 (a_2 - a_4)^4 (a_3 - a_4)^4 & . & . \end{vmatrix} = -\frac{16}{5} \zeta_4^{\frac{1}{2}} \begin{vmatrix} (a_1 - a_2)^5 & (a_1 - a_3)^5 & (a_1 - a_4)^5 \\ (a_2 - a_3)^5 & (a_2 - a_4)^5 \\ (a_3 - a_4)^5 \end{vmatrix}$$

Similar procedure in connection with the theorems of §§1, 9 gives the quotient of two Pfaffians in terms of a zero-axial skew permanent.

Note should also be taken of the theorem derived after the manner of §2, the lines of units now extending, however, to the permanent on the right; for example

$$\begin{vmatrix} (a_1 - a_2)^5 & (a_1 - a_3)^5 & 1 \\ (a_2 - a_3)^5 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{5}{4} \zeta^{\frac{1}{3}} \begin{vmatrix} . & a_1 - a_2 & a_1 - a_3 & 1 \\ a_2 - a_1 & . & a_2 - a_3 & 1 \\ a_3 - a_1 & a_3 - a_2 & . & 1 \\ -1 & -1 & -1 & . \end{vmatrix}.$$

(11) In connection with §4 we had occasion to note that, when the permanents to which we were there led were of odd order, they had the value 0. On investigation it was found to be generally true that *zero-axial skew permanents of odd order vanish*. Probably the simplest proof is gradational. Thus, the theorem manifestly holding when the order is the 3rd, let us examine the case of the 5th order, say the permanent

$$\begin{vmatrix} . & a & b & c & d \\ -a & . & e & f & g \\ -b & -e & . & h & i \\ -c & -f & -h & . & j \\ -d & -g & -i & -j & . \end{vmatrix}.$$

Taking binary products of the elements of the first row and the first column we see at once that the cofactors of $-a^2$, $-b^2$, $-c^2$, $-d^2$ are all 0, and that the cofactor of any other product is the sum of two three-line permanents one of which is the product of the other by $(-1)^3$ and thus cancels that other.

(12) As for zero-axial skew permanents of even order our only incidental result concerns the bordering of

$$\begin{vmatrix} . & . & a & b & c \\ -a & . & d & e & . \\ -b & -d & . & f & . \\ -c & -e & -f & . & . \end{vmatrix}$$

whose value is

$$a^2 f^2 + b^2 e^2 + c^2 d^2 + 2afbe - 2afcd - 2becd$$

or

$$\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix}^2 + 4 \begin{vmatrix} af & cd & be \\ cd & af & be \end{vmatrix},$$

namely, the common element of the bordering lines being 0,

$$\begin{vmatrix} . & l & m & n & r \\ a & . & a & b & c \\ \beta & -a & . & d & e \\ \gamma & -\beta & -d & . & f \\ \delta & -c & -e & -f & . \end{vmatrix} = \left(\begin{vmatrix} a & \beta & \gamma & \delta \\ l & m & n & r \end{vmatrix} \right) \left(fN, eM, dL, cL, bM, aN \right)$$

where

$$\begin{aligned} L &= -af + be + cd \\ M &= af + be + cd \\ N &= af + be + cd. \end{aligned}$$